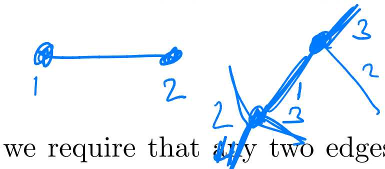
GOOD MORNING

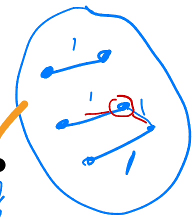


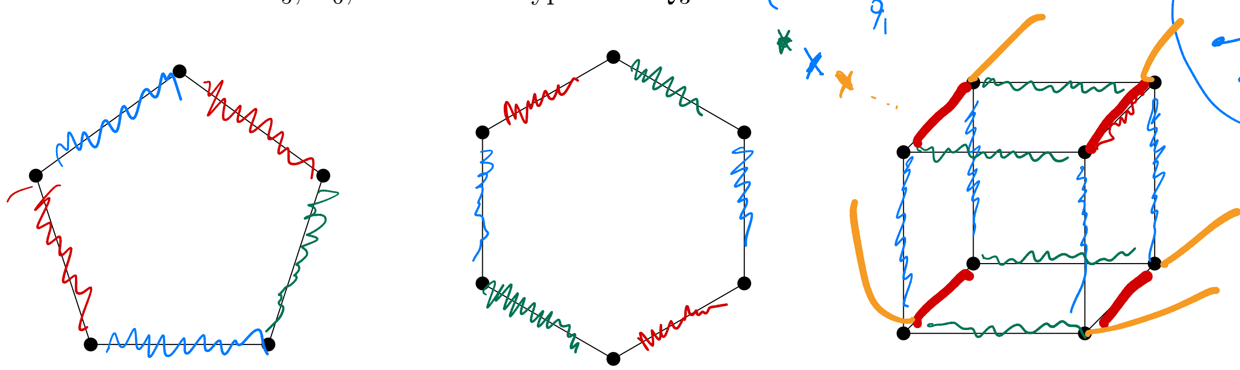
# Lecture 3 - Edge Colorings

As one can color vertices, one can also color the edges of a graph. Here we require that any two edges that share a common end-vertex are colored differently. The smallest number of needed colors to color the edges of a graph $G$ is called the *chromatic index* of $G$, and it is denoted by $\chi'(G)$. Note that each color class induces a matching of the graph.

**1:** Find chromatic index of $C_5$, $C_6$, and the 3D-hypercube $Q_3$.

$(0,1,0,*,0,0,1) \in \{0,1\}^d$



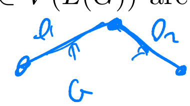


$\chi'(C_5)=3$  $\chi'(C_6)=2$  $\chi'(Q_3)=3$

MATCHING.. 2
EDGES .. 5  $\lceil 5/2 \rceil \le \chi'(C_5)$

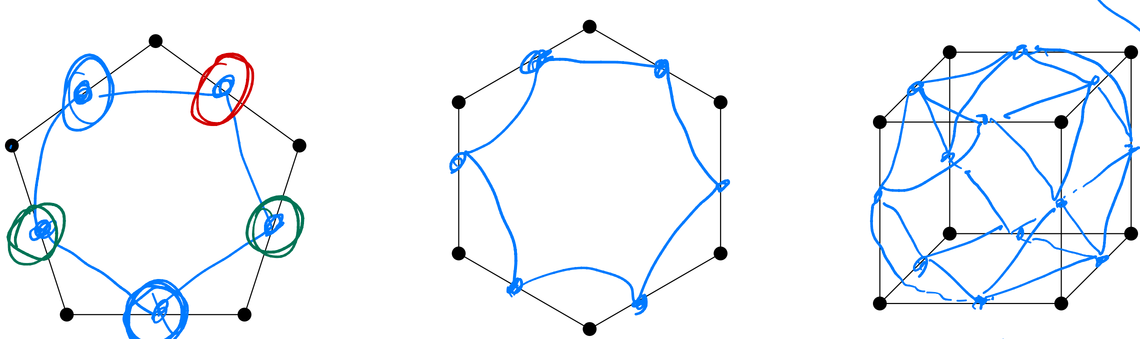
$3 = \Delta(Q_3) \le \chi'(Q_3)$

An edge coloring of a graph $G$ can be considered as a vertex coloring of its line graph $L(G)$. Recall that $V(L(G)) = E(G)$, and two vertices $e, f \in V(L(G))$ are adjacent when edges $e$ in $f$ are incident in $G$. So we have the following claim.

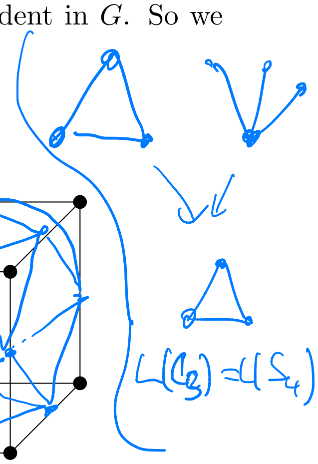


$e_1$ — $e_2$  $L(G)$

**2:** Find line graphs of $C_5$, $C_6$, and $Q_3$.



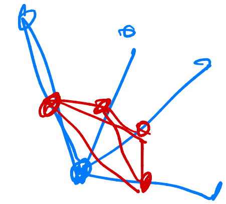
$L(C_5)$  $L(C_6)$  $L(Q_3)$



$L(C_3)=L(S_4)$

**Proposition 1.** *For any graph $G$,*

$$\chi'(G) = \chi(L(G)).$$



**3:** Why is the proposition true?

$\exists$ BIJECTION BETWEEN EDGE-COLORINGS OF $G$ & COLORINGS OF $L(G)$

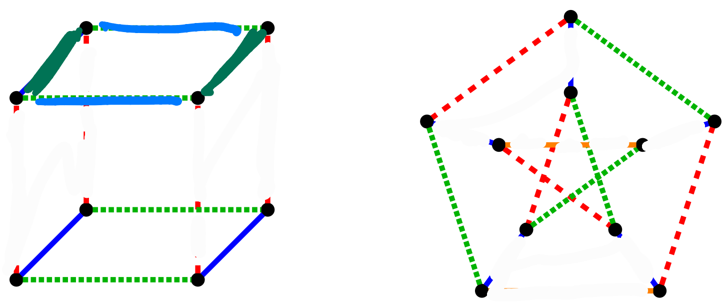
$L(G)$ CONTAINS $\omega(L(G)) \ge \Delta(G)$

Obviously, we need at least $\Delta(G)$ colors to color the edges of $G$, i.e., $\chi'(G) \ge \Delta(G)$. Surprisingly, $\Delta(G)+1$ will be always enough - Vizing's Theorem later.

Notice that $Q_3$ and $C_6$ are bipartite graph, and it has chromatic index as its maximum degree. With the following classical theorem of König from 1916, we will see that this is a case for every bipartite graph.

First, we do the following observation.

**4:** Let $c$ be an edge-coloring of a graph $G$. Let $\alpha$ and $\beta$ be two distinct colors. How does the subgraph of $G$ induced by edges colored $\alpha$ or $\beta$ look like? Denote such subgraph by $H_{\alpha,\beta}$. Explore the following coloring for inspiration.



$\Delta(H_{\alpha,\beta}) \le 2$ .. PATHS, CYCLES,

EVEN!

SWAP COLORS

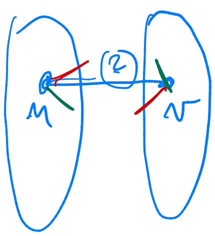
COMPONENT .. KEMPE CHAIN

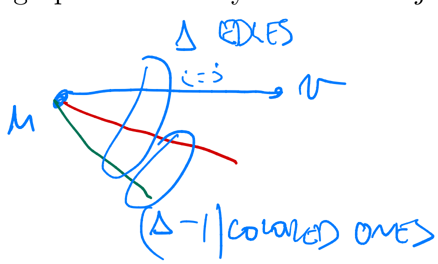
**Theorem 2** (König). *For every bipartite multigraph $G$, it holds*

$$\chi'(G) = \Delta(G).$$

*Proof.* Let $\Delta = \Delta(G)$. Suppose we have colored all the edges of $G$ except edge $e = uv$. As there are at most $\Delta - 1$ colored edges at $u$, there must be a color $i$ not present at $u$. Similarly, there exists a color $j$ not used on the edges of $v$.

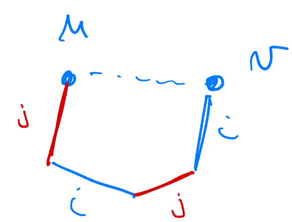
**5:** Look at the subgraphs induced by colors $i$ and $j$ and finish the proof.



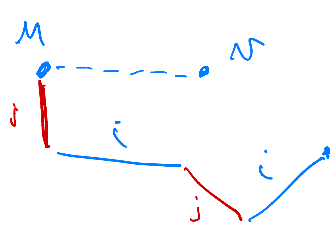

COLOR i LEFT AT u

COLOR j NOT USED AT v



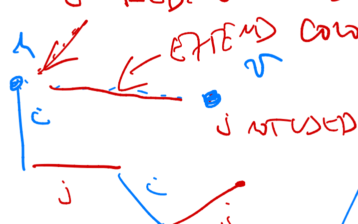

(i≠j) THEN:

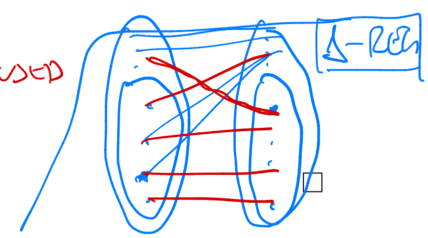
i & ~~RED~~ j




j = RED NOT USED

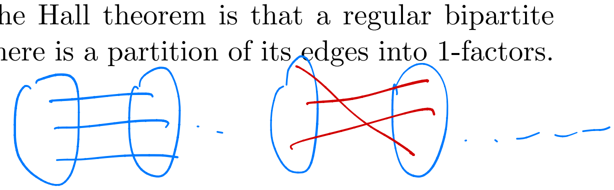
EXTEND COLORING TO uv



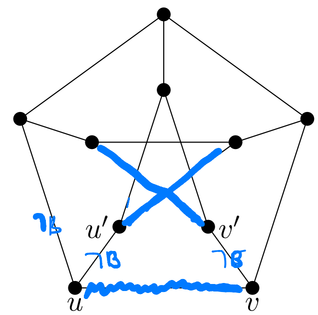

$\square$

One can give an alternative proof in the following way. As an exercise show that any bipartite graph is a subgraph of a bipartite regular graph. An easy consequence of the Hall theorem is that a regular bipartite (multi-)graph has 1-factor, in fact, it is a 1-factorable graph, i.e., there is a partition of its edges into 1-factors. And, these 1-factors induce an edge-coloring of the original graph.



# 1 Vizing's theorem

**6:** Show that the Petersen graph is not 3-edge colorable.

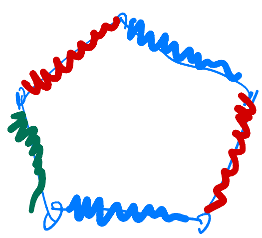



BOTH u' & v' HAVE INCIDENT EDGE COLORED BLUE IF uv IS BLUE & u'v' IS NOT EDGE

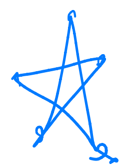
Hint: Suppose for contradiction that there is a 3-edge-coloring. If $uv$ has color $c$, what colors are present at $u'$ and $v'$? What is $\chi(C_5)' =$?.

(1) FOR ANY PERFECT MATCHING M   G−M CONTAINS $C_5$

(2)



ON



ARE ≥ 2 BLUE EDGES
≥ 2 RED EDGES
≥ 2 GREEN EDGES

$|E(C_5)| = 5$

$C_5$ NEEDS ≥ 6 EDGES

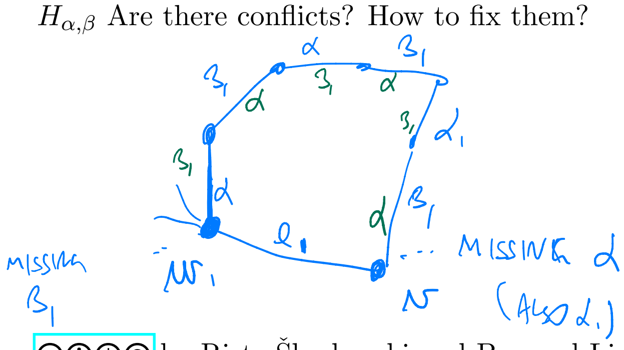
let us introduce a definition. Let $G$ be a properly edge-colored graph and $\alpha, \beta$ two distinct colors used. Observe that the subgraph $H_{\alpha,\beta}$ of $G$ induced by these two colors is comprised of even cycles and paths on which these two colors alternating. Notice that by swapping these two colors on a component of $H_{\alpha,\beta}$, the coloring still stays proper. Actually, we already use this technic in the above theorem. Subgraphs as $H_{\alpha,\beta}$ are called *Kempe chains*, as Kempe was the first to apply them in some arguments (though he did that for vertex colorings).

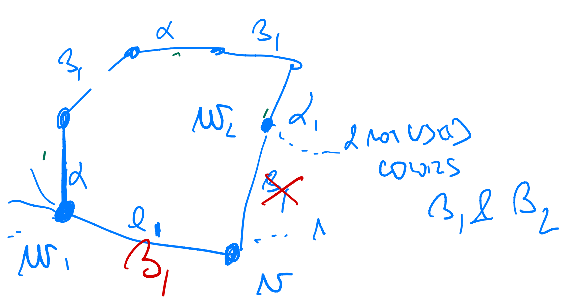
**Theorem 3** (Vizing)**.** *Every simple graph satisfies*

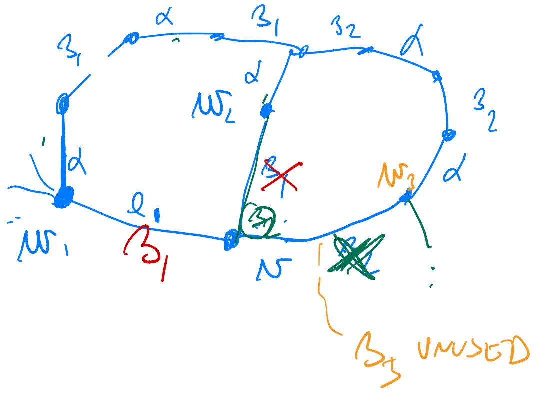
$$\chi'(G) \leq \Delta(G) + 1.$$

*Proof.* Suppose this does not hold for $G$ and let $\Delta = \Delta(G)$. Think of induction on the number of edges if you do not like contradiction with smallest counterexample. We may assume that we have colored all the edges of $G$ but one $e_1 = vw_1$. Since we have $\Delta + 1$ available colors, there is a color missing at $v$, say $\alpha$, and there is a color missing at $w_1$, say $\beta_1$. We may assume that we cannot choose $\alpha$ and $\beta_1$ to be the same color, since we could assign this color to $vw_1$ and get a contradiction.

**7:** Sketch the situation. What happens when you try to modify the coloring to assign $\beta_1$ to $e_1$? Hint - see $H_{\alpha,\beta}$ Are there conflicts? How to fix them?






Repeat this 'shift' over and over again. Means get colors $\beta_1 \neq \beta_2 \neq \beta_3, \ldots$.

**8:** Why can we take $\neq$ in the $\beta_i$ and $\beta_{i+1}$?

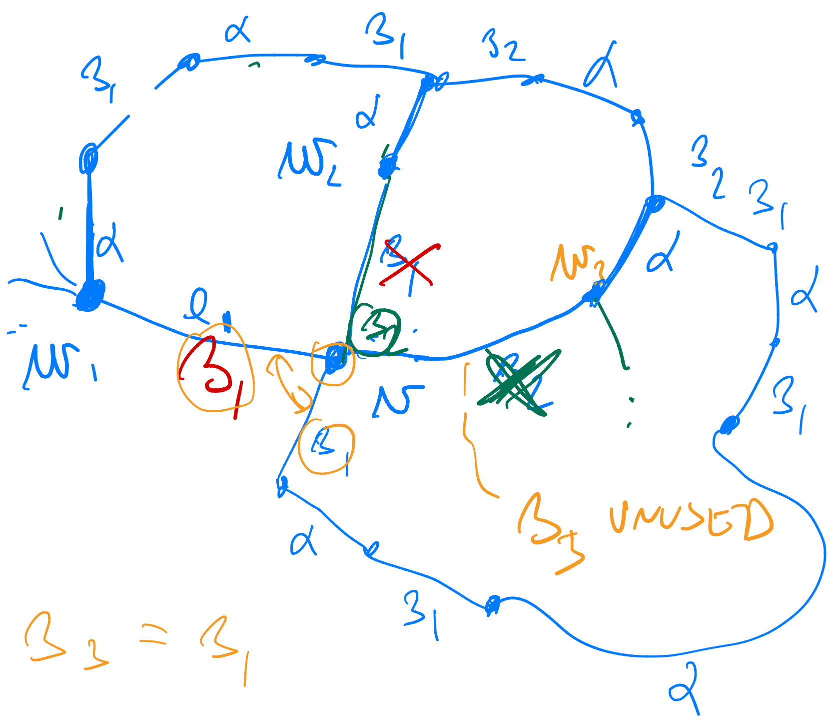
As $\Delta$ is finite, we encounter situation where edge $e_k = vw_k$ is uncolored and at $w_k$ is missing some color $\beta_k$ that satisfies one of the following:

- $\beta_k$ does not show at $v$, or
- $\beta_k = \beta_i$ for some $i < k - 1$, i.e., a color that we encountered before.

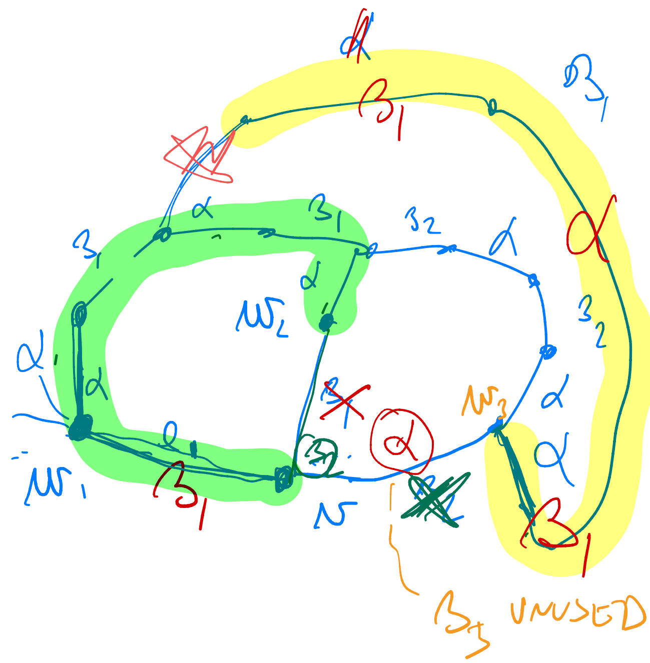
$i = k-1$

**9:** Why will this happen?

**10:** How to finish the proof in either of the two cases?




$\beta_3 = \beta_1$

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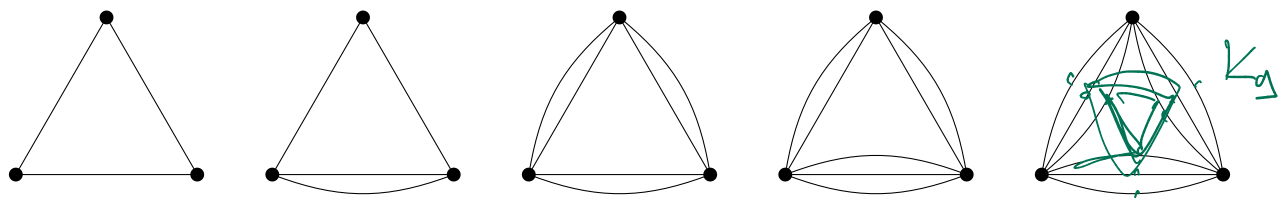
Vizing's theorem arises a very interesting problem. Let

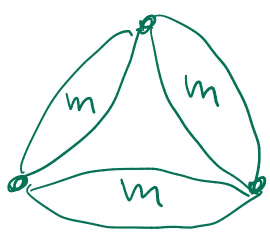
- *Class I* be simple graphs $G$ for which $\Delta(G) = \chi'(G)$,
- *Class II* be simple graphs $G$ for which $\Delta(G) = \chi'(G)$ 1.

Thus, $Q_3$ is a Class I graph and Petersen is a Class II graph. We can ask for every graph is it in Class I or in Class II. From algorithmic point of view, it is NP-complete to decide for a graph which of these two classes is of Holyer. Also worthy to mention that Erdős and Wilson showed that almost all graphs are of Class I.

**11:** Show that Vizing's theorem does not hold for multigraphs. Consider the following graphs, called Shannon triangles.
Generalize the construction and find a constant $c$ such that this construction is showing $\chi'(G) \geq c \cdot \Delta(G)$.




| | | | | | |
|---|---|---|---|---|---|
| $\chi'(G) =$ | 3 | 4 | 6 | 7 | 9 |
| $\Delta$ | 2 | 3 | 4 | 5 | 6 |



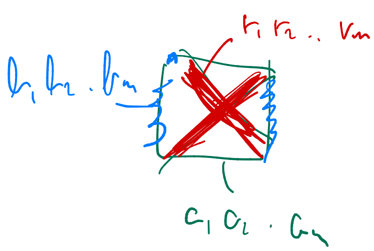

$\Delta = 2m$

$\chi' = 3m$

$\chi'(G) \geq \frac{3}{2}\Delta(G)$

$\chi'(G) = \Delta + \mu$

$2m + m = 3m$



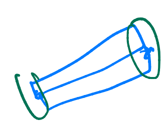

$3m = \chi'$

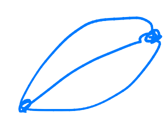
$3m = \Delta$

The following theorem gives an upper bound of $\chi'$ in term of $\Delta$ for multigraphs.

**Theorem 4** (Shannon)**.** *Every graph $G$ satisfies*

$$\chi'(G) \leq \left\lfloor \tfrac{3}{2}\Delta(G) \right\rfloor.$$





The *multiplicity* of a graph $G$, denoted by $\mu(G)$, is the maximum number of edges that are pairwise parallel, i.e., that have both end-vertices the same. Simple graphs have multiplicity 1. Vizing and Gupta independently generalized Theorem 3 to loopless multigraphs involving the multiplicity.

**Theorem 5** (Vizing, Gupta)**.** *Every graph $G$ satisfies*

$$\chi'(G) \leq \Delta(G) + \mu(G).$$

### 1.1 Goldberg conjecture

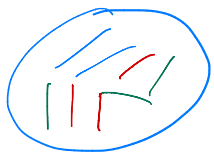
Comparing the last two theorems, Shannon bounds is sharper when $\mu > \Delta/2$, and oppositely for $\mu < \Delta/2$ sharper is the one of Vizing and Gupta. So, combining the last two results for multigraphs we have that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + \min\left\{\mu(G), \lfloor\frac{\Delta(G)}{2}\rfloor\right\}$$

telling us that we have many more possibilities than just two as it is for simple graphs. Let us state a well-known conjecture, which will somehow restrict the possibilities of chromatic index to just three possibilities.

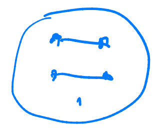
**12:** Suppose we have some optimal edge coloring of $G$. Let $S$ be a subset of vertices of $G$ such that $|S| \geq 3$ is of odd order. Show that

$$\chi'(G) \geq \frac{2|E(G[S])|}{|S|-1} = \frac{|E(G[S])|}{\left(\frac{S-1}{2}\right)} = \frac{|V(L(G[S]))|}{\alpha(L(G[S]))}$$

WHAT IS THE MAX SIZE OF A COLOR CLASS? UPPER BOUND ON # OF BLUE EDGES?

$\boxed{\frac{S-1}{2}}$



S=5

Let

$$\rho(G) = \max\left\{\frac{2|E(G[S])|}{|S|-1} : S \subseteq V(G) \text{ with } S \text{ being odd and of size } \geq 3\right\}. \tag{1}$$

Obviously $\rho(G)$ is a lower bound for $\chi'(G)$. The next conjecture, proposed independently by Goldberg and Seymour is an attempt to preserve the dichotomy of simple graphs to only few case in multigraphs.

**Conjecture 6** (Goldberg, Seymour)**.** *For every multigraph $G$, the chromatic index $\chi'(G)$ equals $\Delta(G)$ or $\Delta(G)+1$ or $\rho(G)$.*